

Scalar extensions for algebraic structures of Łukasiewicz logic

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Abstract

In this paper we study the tensor product for MV-algebras, the algebraic structures of Łukasiewicz ∞ -valued logic. Our main results are: the proof that the tensor product is preserved by the categorical equivalence between the MV-algebras and abelian lattice-order groups with strong unit and the proof of the scalar extension property for semisimple MV-algebras. We explore consequences of this results for various classes of MV-algebras and lattice-ordered groups enriched with a product operation.

Introduction

MV-algebras are the algebraic counterpart of Łukasiewicz ∞ -valued logic [6]. The variety of MV-algebras is generated by the standard model $([0, 1], \oplus, *, 0)$, where $[0, 1]$ is the unit real interval, $x \oplus y = \min\{1, x + y\}$ and $x^* = 1 - x$ for any $x, y \in [0, 1]$. Since the interval $[0, 1]$ is closed to the real product, a natural problem was to analyze the systems obtained by enriching Łukasiewicz logic

with a product operation. This investigation lead to fruitful investigations both in algebra and logic.

Several extension of the notion have been defined by endowing an MV-algebra with products: an internal binary product leads to the notion of PMV-algebra [9]; a scalar product leads to the notion of MV-module [10] and Riesz MV-algebra [11]; a combination of both leads to the notion of f MV-algebra [17]. For all these structures, corresponding logical systems are developed.

Within several important results, one main achievement in the theory is the categorical equivalence with abelian lattice-ordered groups with strong unit [20]. The categorical equivalence for MV-algebras extends naturally extended to any MV-algebra with product, and allows us to connect PMV-algebras, MV-modules, Riesz MV-algebras and f MV-algebras with ℓ -rings with strong unit, ℓ -modules with strong unit, Riesz Spaces with strong unit and f -algebras with strong unit respectively.

The tensor product and its uses are well-known, therefore its definition was given by Martinez for lattice-ordered groups [18] and Mundici for MV-algebras [21]. An important subclass of MV-algebras are the semisimple ones which correspond, through the above mentioned categorical equivalence, to archimedean lattice-ordered structures. Since the tensor product does not preserve the semisimplicity, the semisimple tensor product was defined in [21] for MV-algebras and [16, 5] for lattice-ordered groups.

The scalar extension property (SEP) is one of the basic properties arising from a tensor product; while it is straightforward in the non-ordered case, it presents some difficulties in the framework of lattice-ordered structures. Note that MV-algebras have a natural order wich makes them lattice-ordered structures as well. In Section 3 we give an account of the results known so far, we prove SEP for the semisimple tensor product and we analyze some of its consequences. Since in Section 2 we prove that the tensor product is preserved by the categorical equivalence between MV-algebras and lattice-ordered groups, the results are stated both in the theory of MV-algebras and in the theory of lattice-ordered groups.

The scalar extension property led us to categorical adjunctions between semisimple MV-algebras and semisimple Riesz MV-algebras in Section 3 and between semisimple PMV-algebras and semisimple f MV-algebras in Section 4. In Section 5 we sum up our results and provide an insight on their significance

for Łukasiewicz logic.

1 Preliminaries

1.1 MV-algebras and ℓu -groups

An *MV-algebra* is an algebraic structure $(A, \oplus, *, 0)$, where $(A, \oplus, 0)$ is a commutative monoid, $*$ is an involution and the identity $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$ is satisfied for any $x, y, z \in A$. We further define $x \odot y = (x^* \oplus y^*)^*$ and $1 = 0^*$. An order can be defined on A by setting $x \leq y$ if and only if $x^* \oplus y = 0$; if we set $x \vee y = (x^* \oplus y)^* \oplus y$ and $x \wedge y = (x^* \vee y^*)^*$ then $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice. We refer to [8, 22] for all the unexplained notions concerning MV-algebras.

If A is an MV-algebra we define a partial operation $+$ as follows: for any $x, y \in A$, $x + y$ is defined if and only if $x \leq y^*$ and, in this case, $x + y = x \oplus y$. Note that this operation is cancellative. Assume A, B and C are MV-algebras. A function $\omega : A \rightarrow B$ is *linear* if $a \leq b^*$ implies $\omega(a) \leq \omega(b)^*$ and $\omega(a + b) = \omega(a) + \omega(b)$. Bilinear functions $\beta : A \times B \rightarrow C$ are defined as usual.

Semisimple MV-algebras will play an important role in our development. If A is an MV-algebra and $\text{Rad}(A)$ is the intersection of its maximal ideals, then A is semisimple if and only if A is isomorphic to a separating MV-algebra of $[0, 1]$ -valued continuous functions defined over some compact Hausdorff space [8].

An ℓu -group is a pair (G, u) , where G is an abelian lattice-ordered group [3, 2] and u is a strong unit. If (G, u) is an ℓu -group, then $[0, u]_G = ([0, u], \oplus, *, 0)$ is an MV-algebra, where $[0, u] = \{x \in G \mid 0 \leq x \leq u\}$ and $x \oplus y = u \wedge (x + y)$, $x^* = u - x$ for any $x \in [0, u]$.

If \mathbf{MV} is the category of MV-algebras and \mathbf{auG} is the category of ℓu -groups equipped with morphisms that preserve the strong unit, then one defines a functor $\Gamma : \mathbf{auG} \rightarrow \mathbf{MV}$ by $\Gamma(G, u) = [0, u]_G$ and $\Gamma(h) = h|_{[0, u_1]_{G_1}}$, where (G, u) is an ℓu -group and $h : G_1 \rightarrow G_2$ is a morphism in \mathbf{auG} between (G_1, u_1) and (G_2, u_2) . The functor Γ establishes a categorical equivalence between \mathbf{auG} and \mathbf{MV} [20]. Moreover, through Γ , semisimple MV-algebras correspond to archimedean ℓu -groups.

In the following $\Gamma(\mathbb{R}, 1)$ will be simply denoted $[0, 1]$ (depending on context,

the MV-algebra structure will be tacitly assumed). By Chang's completeness theorem [7], the variety of MV-algebras is generated by $[0, 1]$.

1.2 MV-algebras endowed with a product operation

Product MV-algebras (PMV-algebras for short) have been defined in [9] in the general case and in [19] an equivalent axiomatization was provide for the unital and commutative structures. A *unital PMV-algebra* is a structure $(P, \oplus, \cdot, *, 0)$ such that $(P, \oplus, *, 0)$ is an MV-algebra and $\cdot : P \times P \rightarrow P$ is a bilinear function such that $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ and $a \cdot 1 = 1 \cdot a = a$ for any $a, b, c \in P$.

A further extension of the notion of MV-algebra has been introduced in [10]. If P is a PMV-algebra, then an *MV-module* over P (*P-MV-module*) is a structure $(M, \oplus, *, \{\alpha | \alpha \in P\}, 0)$ such that $(M, \oplus, *, 0)$ is an MV-algebra, $\{\alpha | \alpha \in P\}$ is a family of unary operations such that the function $(\alpha, x) \mapsto \alpha x$ is bilinear, $(\alpha \cdot \beta)x = \alpha(\beta x)$ and $1x = x$ for any $\alpha, \beta \in P$ and any $x \in M$.

Note that [10, Section 6.4] provides an equational characterization for these structures. Most important for our development is the case $P = [0, 1]$. The MV-modules over $[0, 1]$ are called *Riesz MV-algebras* are studied in [11].

Finally, *unital fMV-algebras* have been introduced in [17] and they are algebraic structures $(A, \oplus, *, \cdot, \{\alpha\}_{\alpha \in [0, 1]}, 0)$ such that $(A, \oplus, *, \cdot, 0)$ is a unital PMV-algebra, $(A, \oplus, *, \{\alpha\}_{\alpha \in [0, 1]}, 0)$ is a Riesz MV-algebra and the condition $\alpha(x \cdot y) = (\alpha x) \cdot y = x \cdot (\alpha y)$ is satisfied for any $\alpha \in [0, 1]$ and $x, y \in A$.

We defined a hierarchy of algebraic structures, all of them having an MV-algebra reduct. Hence there are forgetful functors from the categories **PMV** of PMV-algebras, **RMV** of Riesz MV-algebras and **fMV** of fMV-algebras to **MV**. For each of this structures one can prove a categorical equivalence with an appropriate class of unital lattice-ordered structures having a lattice-ordered group reduct with a strong unit [9, 11, 17]. If **uR** is the category of unital f -rings with strong unit (fu -rings), **uRS** is the category of Riesz spaces with strong unit and **fuAlg** is the category of unital f -algebras with strong unit (fu -algebras), then the categorical equivalence are presented in the following diagram, in which all horizontal arrows are suitable forgetful functors.

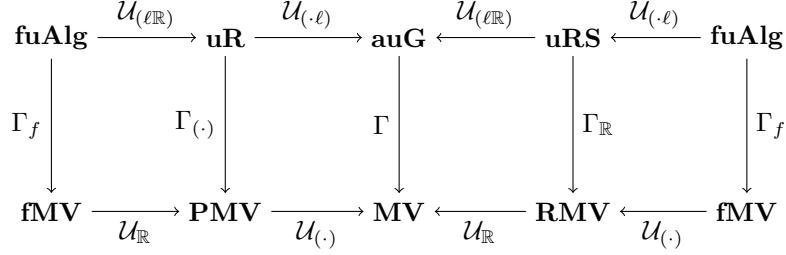


Figure 1.

Note that for the objects in \mathbf{uR} and \mathbf{fuAlg} have a unital ring reduct and we ask, in addition, that the ring-unity coincides with the strong unit. For the general theory of ℓ -rings and f -rings, Riesz spaces, ℓ -algebras and f -algebras we refer to [2, 3, 4, 5, 24].

We note that a PMV-algebra (Riesz MV-algebra, f MV-algebra) is semisimple if its MV-algebra reduct is a semisimple MV-algebra and that semisimple PMV-algebras (semisimple Riesz MV-algebras, semisimple f MV-algebras) correspond to archimedean ℓu -rings (archimedean unital Riesz spaces, archimedean $f u$ -algebras). Also, any unital and semisimple PMV-algebra (f MV-algebra) is commutative by the general theory of f -rings (f -algebras) [4, 24].

In a similar manner, the MV-modules are categorically equivalent to appropriate classes of ℓ -modules with strong unit (ℓu -rings). We refer to [23] for the general theory of ℓ -modules. If (R, u) is an ℓu -ring and $P \simeq \Gamma_{(\cdot)}(R, u)$ then the category of MV-modules over P is equivalent to the category of ℓu -modules over R [10].

1.3 The tensor product \otimes_{MV} and the semisimple tensor product \otimes

A bimorphism is a bilinear function that is \vee -preserving and \wedge -preserving in each component. Bimorphisms were defined in [21], where the additional requirement $\beta(1, 1) = 1$ was imposed. In the present approach we eliminate this restriction.

The *interval algebra* of A is the MV-algebra $[0, a] = \{x \in A \mid 0 \leq x \leq a\}$, endowed with the following operations $x \oplus_a y = (x \oplus y) \wedge a$, $x^{*a} = x^* \odot a$ for any $x, y \in [0, a]$ [21]. In the follow we will use the notation $[0, a] \leq_i A$ in order to say that $[0, a]$ is an interval algebra of A .

The MV-algebraic tensor product was defined in [21] as a universal bimorphism. We shall use in the sequel a slightly modified universal property proved

in [14]. For two MV-algebras A and B , let $A \otimes_{MV} B$ be the tensor product and $\beta_{A,B} : A \times B \rightarrow A \otimes_{MV} B$ the universal bimorphism. Then the following universal property holds:

for any MV-algebra C and for any bimorphism $\beta : A \times B \rightarrow C$, there is a unique homomorphism of MV-algebras $\omega : A \otimes_{MV} B \rightarrow [0, \beta(1, 1)] \leq_i C$ such that $\omega \circ \beta_{A,B} = \beta$.

For $a \in A$ and $B \in B$ we denote $a \otimes_{MV} b = \beta_{A,B}(a, b)$. Note that $A \otimes_{MV} B$ is generated by $\beta_{A,B}(A \times B)$.

In [21] the author proves that there exists a semisimple MV-algebra A such that $A \otimes_{MV} A$ is not semisimple. Therefore he defines the semisimple tensor product of A and B , semisimple MV-algebras, by

$$A \otimes B = A \otimes_{MV} B / \text{Rad}(A \otimes_{MV} B).$$

For the semisimple tensor product the universal property holds with respect to semisimple MV-algebras.

The following representation theorem is crucial for our development.

Theorem 1.1. *[21] Let A and B be semisimple MV-algebras, and let X, Y be the set such that $A \subseteq C(X)$ and $B \subseteq C(Y)$. Let $\gamma : A \times B \rightarrow C(X \times Y)$ be the map defined by $\gamma(a, b)(x, y) = a(x)b(y)$. Then γ is a bimorphism and $A \otimes B$ is isomorphic to the MV-subalgebra of $C(X \times Y)$ generated by $\gamma(a, b)$, with $a \in A$ and $b \in B$.*

Few types of tensor products are defined in the literature of partially-ordered and lattice-ordered groups [18, 23]. We recall the one that will be used in the sequel.

Let G , H and L abelian lattice-ordered groups. An ℓ -bilinear function is a map $\gamma : G \times H \rightarrow L$ such that $\gamma(x, \cdot)$ and $\gamma(\cdot, y)$ are homomorphisms of ℓ -groups when x and y are positive elements in G and H , respectively. Hence, the tensor product $G \otimes_{\ell} H$ and its universal ℓ -bilinear function $\gamma_{G,H} : G \times H \rightarrow G \otimes_{\ell} H$ satisfy a universal property with respect to abelian lattice-ordered groups and ℓ -bilinear functions. For $x \in G$ and $y \in H$ we denote $x \otimes_{\ell} y = \gamma_{G,H}(x, y)$.

In [5, 16] the authors provide a construction for the tensor product of archimedean ℓ -groups, denoted by \otimes_a and they prove the universal property with respect to archimedean structures.

2 The Γ -functor and the tensor product \otimes_{MV}

In this section we prove that the functor $\Gamma : \mathbf{auG} \rightarrow \mathbf{MV}$ commutes with the tensor product, both in the general and in the archimedean case. To do this, we prove an extension result for bimorphisms.

As a preliminary step, we give the detailed proof of the fact that any factor of the MV-algebraic tensor product is embedded in the tensor product. If A and B are MV-algebras, then we define

$$\begin{aligned} \iota_A : A &\rightarrow A \otimes_{MV} B \text{ and } \iota_B : B \rightarrow A \otimes_{MV} B \\ \iota_A(a) &= a \otimes_{MV} 1_B \text{ and } \iota_B(b) = 1_A \otimes_{MV} b \text{ for any } a \in A \text{ and } b \in B. \end{aligned}$$

The functions ι_A and ι_B are embeddings of MV-algebras [13] (private communication). We sketch the proof, for the sake of completeness.

Proposition 2.1. *The maps $\iota_A : A \rightarrow A \otimes_{MV} B$ and $\iota_B : B \rightarrow A \otimes_{MV} B$ defined as $\iota_A(a) = a \otimes_{MV} 1_B$ and $\iota_B(b) = 1_A \otimes_{MV} b$ for any $a \in A$ and $b \in B$, are embeddings.*

Proof. By [22, Theorem 2.20] there exists a MV-algebra D such that both A and B embeds in it. By [8, Theorem 9.5.1] there exists a set X and a MV-algebra embedding $f : D \hookrightarrow (*[0, 1])^X$, therefore there exist two embeddings $A \xrightarrow{f_A} (*[0, 1])^X$ and $B \xrightarrow{f_B} (*[0, 1])^X$. We remark that $(*[0, 1])^X$ is a unital and commutative PMV-algebra, therefore we define the following bimorphism

$$\beta : A \times B \rightarrow (*[0, 1])^X, \beta_1(a, b) = f_A(a) \cdot f_B(b) \text{ for any } a \in A \text{ and any } b \in B.$$

By the universal property in [21], there exists $\omega : A \otimes_{MV} B \rightarrow (*[0, 1])^X$ such that $\omega(a \otimes_{MV} b) = f_A(a) \cdot f_B(b)$.

Assume that $\iota_A(a_1) = \iota_A(a_2)$, that is $a_1 \otimes_{MV} 1_B = a_2 \otimes_{MV} 1_B$, then $f_A(a_1) = \omega(a_1 \otimes_{MV} 1_B) = \omega(a_2 \otimes_{MV} 1_B) = f_A(a_2)$. Since f_A is an embedding, the conclusion follows. Analogously, ι_B is an embedding. \square

Assume (G, u_G) (H, u_H) and (L, u_L) are ℓu -groups and $\gamma : G \times H \rightarrow L$ is an ℓ -bilinear function. We say that γ is ℓu -bilinear if $\gamma(u_G, u_H) \leq u_L$.

In the sequel we prove that a bimorphism uniquely extends to an ℓu -bilinear function.

Proposition 2.2. *If $A = \Gamma(G, u_G)$, $B = \Gamma(H, u_H)$ and $C = \Gamma(L, u_L)$ and $\beta : A \times B \rightarrow C$ is a bimorphism, then there exists an unique ℓu -bilinear function $\bar{\beta} : G \times H \rightarrow L$ that extends β .*

Proof. Let a be a fixed element in A and denote $\beta_a = \beta(a, \cdot)$. Hence, by [21, Proposition 2.3], we have $\beta_a : B \rightarrow [0, \beta(a, 1)] \leq_i C$ is a homomorphism of MV-algebras. By [14, Proposition 2.9], there exists a unique homomorphism of ℓ -groups $\overline{\beta_a} : H \rightarrow L$, and $\overline{\beta_a}|_B = \beta_a$.

Step 1. We prove that the map $\gamma_h : A \rightarrow L$, defined by $\gamma_h(a) = \overline{\beta_a}(h)$ for any $h \in H$, is linear.

Let the sum $a_1 + a_2$ be defined in A and let $h \in H$, since B generates the positive cone of (H, u_H) , $h = h^+ - h^-$, where $h^+ = s_1 + \dots + s_n$ and $h^- = t_1 + \dots + t_m$, with $s_i, t_j \in B$ for any $i = 1, \dots, n$ and $j = 1, \dots, m$.

$$\begin{aligned} \gamma_h(a_1 + a_2) &= \overline{\beta_{a_1+a_2}}((s_1 + \dots + s_n) - (t_1 + \dots + t_m)) = \\ &= (\overline{\beta_{a_1+a_2}}(s_1) + \dots + \overline{\beta_{a_1+a_2}}(s_n)) - (\overline{\beta_{a_1+a_2}}(t_1) + \dots + \overline{\beta_{a_1+a_2}}(t_m)) = (\beta(a_1+a_2, s_1) + \\ &+ \dots + \beta(a_1+a_2, s_n)) - (\beta(a_1+a_2, t_1) + \dots + \beta(a_1+a_2, t_m)) = (\beta(a_1, s_1) + \beta(a_2, s_1) + \\ &+ \dots + \beta(a_1, s_n) + \beta(a_2, s_n)) - (\beta(a_1, t_1) + \beta(a_2, t_1) + \dots + \beta(a_1, t_m) + \beta(a_2, t_m)) = \\ &= [(\overline{\beta_{a_1}}(s_1) + \dots + \overline{\beta_{a_1}}(s_n)) - (\overline{\beta_{a_1}}(t_1) + \dots + \overline{\beta_{a_1}}(t_m))] + \\ &+ [(\overline{\beta_{a_2}}(s_1) + \dots + \overline{\beta_{a_2}}(s_n)) - (\overline{\beta_{a_2}}(t_1) + \dots + \overline{\beta_{a_2}}(t_m))] = \overline{\beta_{a_1}}(h) + \overline{\beta_{a_2}}(h) = \\ &= \gamma_h(a_1) + \gamma_h(a_2). \end{aligned}$$

Step 2. γ_h commutes with \wedge and \vee .

We first remark that an element of (H, u_H) is a good sequence according with Mundici construction of the inverse of the functor Γ . Moreover, by [1] and it is possible to take indexes in \mathbb{Z} instead of quotients of sequence, then

$$\begin{aligned} \gamma_h(a_1 \wedge a_2) &= \overline{\beta_{a_1 \wedge a_2}}(h) = \overline{\beta_{a_1 \wedge a_2}}((h_i)_{i \in \mathbb{Z}}) = (\overline{\beta_{a_1 \wedge a_2}}(h_i))_{i \in \mathbb{Z}} = \\ &= (\beta(a_1 \wedge a_2, h_i))_{i \in \mathbb{Z}} = (\beta(a_1, h_i) \wedge \beta(a_2, h_i))_{i \in \mathbb{Z}} = (\beta(a_1, h_i))_{i \in \mathbb{Z}} \wedge (\beta(a_2, h_i))_{i \in \mathbb{Z}} = \\ &= (\overline{\beta_{a_1}}(h_i))_{i \in \mathbb{Z}} \wedge (\overline{\beta_{a_2}}(h_i))_{i \in \mathbb{Z}} = \overline{\beta_{a_1}}(h) \wedge \overline{\beta_{a_2}}(h) = \gamma_h(a_1) \wedge \gamma_h(a_2), \end{aligned}$$

and similarly for \vee .

Therefore the map $\gamma_h : [0, u_G] \rightarrow L$ is linear and commutes with \vee and \wedge , i.e. $\gamma_h : [0, u_G] \rightarrow [0, \gamma_h(u_G)] \leq_i [0, u_L]$ is an homomorphism of MV-algebras [21, Proposition 2.3] and consequently, by [14, Proposition 2.9] there exists a unique homomorphism of ℓ -groups $\overline{\gamma_h} : G \rightarrow K \leq L$, where K is the ℓ -group generated by $\gamma_h(A)$ and $\overline{\gamma_h}|_A = \gamma_h$.

We define now $\overline{\beta} : G \times H \rightarrow L$ as $\overline{\beta}(g, h) = \overline{\gamma_h}(g)$. By the construction, $\overline{\beta}(\cdot, h)$ is a homomorphism of ℓ -groups.

Step 3. $\overline{\beta}(g, \cdot)$, with g fixed element in G , is linear.

Let h_1, h_2 be elements in H ; there exist suitable elements in the unit interval such that $g^+ = s_1 + \dots + s_n$ and $g^- = t_1 + \dots + t_m$.

$$\begin{aligned}
\overline{\beta}(g, h_1 + h_2) &= \overline{\gamma_{h_1+h_2}}(g) = \\
&= (\overline{\gamma_{h_1+h_2}}(s_1) + \dots + \overline{\gamma_{h_1+h_2}}(s_n)) - (\overline{\gamma_{h_1+h_2}}(t_1) + \dots + \overline{\gamma_{h_1+h_2}}(t_m)) = \\
&= (\overline{\beta_{s_1}}(h_1 + h_2) + \dots + \overline{\beta_{s_n}}(h_1 + h_2)) - (\overline{\beta_{t_1}}(h_1 + h_2) + \dots + \overline{\beta_{t_m}}(h_1 + h_2)) = \\
&= (\overline{\beta_{s_1}}(h_1) + \overline{\beta_{s_1}}(h_2) + \dots + \overline{\beta_{s_n}}(h_1) + \overline{\beta_{s_n}}(h_2)) - (\overline{\beta_{t_1}}(h_1) + \overline{\beta_{t_1}}(h_2) + \dots + \\
&= \overline{\beta_{t_m}}(h_1) + \overline{\beta_{t_m}}(h_2)) = [(\gamma_{h_1}(s_1) + \dots + \gamma_{h_1}(s_n)) - (\gamma_{h_1}(t_1) + \dots + \gamma_{h_1}(t_m))] + \\
&= [(\gamma_{h_2}(s_1) + \dots + \gamma_{h_2}(s_n)) - (\gamma_{h_2}(t_1) + \dots + \gamma_{h_2}(t_m))] = \overline{\gamma_{h_1}}(g) + \overline{\gamma_{h_2}}(g) = \\
&= \overline{\beta}(g, h_1) + \overline{\beta}(g, h_2).
\end{aligned}$$

Step 4. $\overline{\beta}(g, \cdot)$ commute with \vee and \wedge .

We will use again good sequences

$$\begin{aligned}
\overline{\beta}(g, h_1 \wedge h_2) &= \overline{\gamma_{h_1 \wedge h_2}}(g) = \overline{\gamma_{h_1 \wedge h_2}}((g_i)_{i \in \mathbb{Z}}) = (\overline{\gamma_{h_1 \wedge h_2}}(g_i))_{i \in \mathbb{Z}} = \\
&= (\gamma_{h_1 \wedge h_2}(g_i))_{i \in \mathbb{Z}} = (\overline{\beta_{g_i}}(h_1 \wedge h_2))_{i \in \mathbb{Z}} = (\overline{\beta_{g_i}}(h_1) \wedge \overline{\beta_{g_i}}(h_2))_{i \in \mathbb{Z}} = (\overline{\beta_{g_i}}(h_1))_{i \in \mathbb{Z}} \wedge \\
&= (\overline{\beta_{g_i}}(h_2))_{i \in \mathbb{Z}} = (\overline{\gamma_{h_1}}(g_i))_{i \in \mathbb{Z}} \wedge (\overline{\gamma_{h_2}}(g_i))_{i \in \mathbb{Z}} = \overline{\gamma_{h_1}}(g) \wedge \overline{\gamma_{h_2}}(g) = \overline{\beta}(g, h_1) \wedge \overline{\beta}(g, h_2).
\end{aligned}$$

The same can be done for \vee and $\overline{\beta}(g, \cdot)$ is a homomorphism of ℓ -groups. Moreover, $\overline{\beta}(u_G, u_H) = \beta(u_G, u_H) \leq u_L$.

In order to prove the uniqueness, we assume that $\tilde{\beta} : G \times H \rightarrow L$ is another ℓu -bilinear function that extends β . If $a \in A$ then $\tilde{\beta}(a, \cdot)$ is an extension of β_a , so it coincides with $\overline{\beta}_a$. It follows that $\overline{\beta}(a, \cdot) : H \rightarrow L$ and $\tilde{\beta}(a, \cdot) : H \rightarrow L$ coincide for any $a \in A$. By linearity, they coincide for any $g \in G$. \square

The main result of this section is Theorem 2.1, which asserts that the functor $\Gamma : \mathbf{auG} \rightarrow \mathbf{MV}$ preserves the tensor product.

Note that, if (G, u_G) and (H, u_H) are ℓu -groups then $u_G \otimes_\ell u_H$ is strong unit in $G \otimes_\ell H$ [18, 3.6]. In the sequel we prove two preliminary lemmas.

Lemma 2.1. *Let (G, u_G) , (H, u_H) and (L, u_L) be ℓu -groups. For any bimorphism $\gamma : \Gamma(G, u_G) \times \Gamma(H, u_H) \rightarrow \Gamma(L, u_L)$ there is a unique homomorphism of MV-algebras $\omega : \Gamma(G \otimes_\ell H, u_G \otimes_\ell u_H) \rightarrow [0, \gamma(u_G, u_H)] \leq_i \Gamma(L, u_L)$ such that $\omega(x \otimes_\ell y) = \gamma(x, y)$ for any $x \in \Gamma(G, u_G)$ and $y \in \Gamma(H, u_H)$.*

Proof. We set $A = \Gamma(G, u_G)$, $B = \Gamma(H, u_H)$, $C = \Gamma(L, u_L)$ and we suppose that $\gamma : A \times B \rightarrow C$ is a bimorphism. By Proposition 2.2, there is a ℓu -bilinear function $\tilde{\gamma} : G \times H \rightarrow L$ which extends γ , so there is a unique homomorphism of ℓ -groups $\tilde{\omega} : G \otimes_\ell H \rightarrow L$ such that $\tilde{\omega} \circ \gamma_{G,H} = \tilde{\gamma}$.

It follows that $\tilde{\omega}(u_G \otimes_\ell u_H) = \tilde{\gamma}(u_G, u_H) = \gamma(1, 1) \leq u_L$, therefore by [14, Lemma 2.8] the restriction $\omega : \Gamma(G \otimes_\ell H, u_G \otimes_\ell u_H) \rightarrow [0, \gamma(u_G, u_H)] \leq_i C$, defined by $\omega(\mathbf{x}) = \tilde{\omega}(\mathbf{x})$ for any $\mathbf{x} \in \Gamma(G \otimes_\ell H, u_G \otimes_\ell u_H)$, is a homomorphism

of MV-algebras and $\omega(x \otimes_\ell y) = \tilde{\omega}(x \otimes_\ell y) = (\tilde{\omega} \circ \gamma_{G,H})(x, y) = \tilde{\gamma}(x, y) = \gamma(x, y)$, for any $x \in \Gamma(G, u_G)$ and $y \in \Gamma(H, u_H)$.

In order to prove the uniqueness, let $\theta : \Gamma(G \otimes_\ell H, u_{G \otimes_\ell H}) \rightarrow [0, \gamma(u_G, u_H)] \leq_i C$ be another homomorphism of MV-algebras such that $\theta(x \otimes_\ell y) = \gamma(x, y)$ for any $x \in \Gamma(G, u_G)$ and $y \in \Gamma(H, u_H)$. By [14, Proposition 2.9] there is a unique homomorphism of ℓ -groups $\tilde{\theta} : G \otimes H \rightarrow L$ such that $\tilde{\theta}|_{\Gamma(G \otimes_\ell H, u_{G \otimes_\ell H})} = \theta$. If τ is defined as $\tilde{\theta} \circ \gamma_{G,H}$ then is straightforward that $\tau : G \times H \rightarrow L$ is a ℓu -bilinear function and $\tau(x, y) = \theta(x \otimes y) = \gamma(x, y)$. Since $\tilde{\gamma}$ is the unique ℓu -bilinear function that extends γ we get $\tau = \tilde{\gamma}$ and $\tilde{\theta} \circ \gamma_{G,H} = \tilde{\gamma}$. It follows that $\tilde{\theta} = \tilde{\omega}$ and $\theta = \omega$. \square

In the following, we introduce a notation. Let A, B be MV-algebras and $(G_A, u_A), (G_B, u_B)$ ℓu -groups such that $A \simeq \Gamma(G_A, u_A)$ and $B \simeq \Gamma(G_B, u_B)$. Assume $\{\eta_A\}_{A \in \mathbf{MV}}$ is the natural isomorphism between the categories \mathbf{MV} and \mathbf{auG} , i.e. $\eta_A : A \rightarrow \Gamma(G_A, u_A)$ and $\eta_B : B \rightarrow \Gamma(G_B, u_B)$ are isomorphisms of MV-algebra. Hence we define $\gamma_{A,B} : A \times B \rightarrow \Gamma(G_A \otimes_\ell G_B, u_A \otimes_\ell u_B)$ by $\gamma_{A,B}(x, y) = \eta_A(x) \otimes_\ell \eta_B(y)$ for any $x \in A$ and $y \in B$.

Lemma 2.2. *Let A, B and $(G_A, u_A), (G_B, u_B)$ ℓu -groups such that $A \simeq \Gamma(G_A, u_A)$ and $B \simeq \Gamma(G_B, u_B)$. For any MV-algebra C and any bimorphism $\gamma : A \times B \rightarrow C$ there is a unique homomorphism of MV-algebras $\omega : \Gamma(G_A \otimes_\ell G_B, u_A \otimes_\ell u_B) \rightarrow [0, \gamma(1_A, 1_B)] \leq_i C$ such that $\omega \circ \gamma_{A,B} = \gamma$.*

Proof. In the following $A \otimes_\ell B$ will denote the MV-algebra $\Gamma(G_A \otimes_\ell G_B, u_A \otimes_\ell u_B)$. Suppose that C is an arbitrary MV-algebra and $\gamma : A \times B \rightarrow C$ is a bimorphism. We define $\gamma_1 : \Gamma(G_A, u_A) \times \Gamma(G_B, u_B) \rightarrow \Gamma(G_C, u_C)$ by

$$\gamma_1(x, y) = \eta_C(\gamma(\eta_A^{-1}(x), \eta_B^{-1}(y))).$$

Since η_C, η_A^{-1} and η_B^{-1} are MV-algebra isomorphisms, γ_1 is also a bimorphism, and by Lemma 2.1, there is a unique homomorphism of MV-algebras $\omega_1 : \Gamma(G_A \otimes_\ell G_B, u_A \otimes_\ell u_B) \rightarrow [0, \gamma_1(u_A, u_B)] \leq_i \Gamma(G_C, u_C)$ such that $\omega_1(x \otimes_\ell y) = \gamma_1(x, y)$ for any $x \in \Gamma(G_A, u_A)$ and $y \in \Gamma(G_B, u_B)$. Remark that the definition domain of ω_1 is $A \otimes_\ell B$. Hence, if we define $\omega(\mathbf{x}) = \eta_C^{-1}(\omega_1(\mathbf{x}))$ for any $\mathbf{x} \in A \otimes_\ell B$, by [14, Lemma 2.8] $\omega : A \otimes_\ell B \rightarrow [0, \eta_C^{-1}(\gamma_1(u_A, u_B))] \leq_i C$ is a homomorphism of MV-algebras. We have $\eta_C^{-1}(\gamma_1(u_A, u_B)) = \gamma(\eta_A^{-1}(u_A), \eta_B^{-1}(u_B)) = \gamma(1_A, 1_B)$.

For any $x \in A$ and $y \in B$ it follows that

$$(\omega \circ \gamma_{A,B})(x, y) = \omega(\eta_A(x) \otimes_\ell \eta_B(y)) = \eta_C^{-1}(\omega_1(\eta_A(x) \otimes_\ell \eta_B(y))) =$$

$$= \eta_C^{-1}(\gamma_1(\eta_A(x), \eta_B(y))) = \gamma(x, y).$$

Hence, $\omega \circ \gamma_{A,B} = \gamma$. In order to prove the uniqueness, suppose that $\theta : A \otimes_\ell B \rightarrow [0, \gamma(1_A, 1_B)] \leq_i C$ is another homomorphism of MV-algebras such that $\theta \circ \gamma_{A,B} = \gamma$. Using the isomorphism η_C we define the following homomorphism of MV-algebras: $\theta_1 : A \otimes_\ell B \rightarrow [0, \eta_C(\gamma(1_A, 1_B))] \leq \Gamma(G_C, u_C)$, with $\theta_1(\mathbf{x}) = \eta_C(\theta(\mathbf{x}))$, for any $\mathbf{x} \in A \otimes_\ell B$. Remark that

$$\eta_C(\gamma(1_A, 1_B)) = \eta_C(\gamma(\eta_A^{-1}(u_A), \eta_B^{-1}(u_B))) = \gamma_1(u_A, u_B).$$

Moreover, for any $x \in \Gamma(G_A, u_A)$ and $y \in \Gamma(G_B, u_B)$ we get

$$\begin{aligned} \theta_1(x \otimes_\ell y) &= \eta_C(\theta(x \otimes_\ell y)) = \eta_C(\theta(\eta_A(\eta_A^{-1}(x)) \otimes_\ell \eta_B(\eta_B^{-1}(y)))) = \\ &= \eta_C(\theta(\gamma_{A,B}(\eta_A^{-1}(x), \eta_B^{-1}(y)))) = \eta_C(\gamma(\eta_A^{-1}(x), \eta_B^{-1}(y))) = \gamma_1(x, y). \end{aligned}$$

It follows that θ_1 satisfies the properties that uniquely characterize ω_1 by Lemma 2.1, so $\theta_1 = \omega_1$. In consequence,

$$\omega(\mathbf{x}) = \eta_C^{-1}(\omega_1(\mathbf{x})) = \eta_C^{-1}(\theta_1(\mathbf{x})) = \theta(\mathbf{x})$$

for any $\mathbf{x} \in A \otimes_\ell B$, so $\theta = \omega$. \square

We are now ready to prove the main result.

Theorem 2.1. *If (G_A, u_A) , (G_B, u_B) are ℓu -groups and A, B are MV-algebras such that $A \simeq \Gamma(G_A, u_A)$ and $B \simeq \Gamma(G_B, u_B)$ then $A \otimes_{MV} B \simeq \Gamma(G_A \otimes_\ell G_B, u_A \otimes_\ell u_B)$.*

Proof. As before, $A \otimes_\ell B$ denotes $\Gamma(G_A \otimes_\ell G_B, u_A \otimes_\ell u_B)$ and we prove that $A \otimes_{MV} B \simeq A \otimes_\ell B$. Let $\gamma_{A,B} : A \times B \rightarrow A \otimes_\ell B$ the bimorphism defined as in Lemma 2.2 and $\beta_{A,B} : A \times B \rightarrow A \otimes_{MV} B$ the standard bimorphism of the MV-algebraic tensor product. Using universal property of the tensor product, we get a homomorphism of MV-algebras

$$\lambda : A \otimes_{MV} B \rightarrow [0, \gamma_{A,B}(1_A, 1_B)] \leq_i A \otimes_\ell B$$

such that $\lambda_{A,B} \circ \beta_{A,B} = \gamma_{A,B}$. By Lemma 2.2 there exists an homomorphism of MV-algebras

$$\delta : A \otimes_\ell B \rightarrow [0, \beta_{A,B}(1_A, 1_B)] \leq_i A \otimes_{MV} B$$

such that $\delta \circ \gamma_{A,B} = \beta_{A,B}$.

Then we get:

$$(\delta \circ \lambda) \circ \beta_{A,B} = \delta \circ (\lambda \circ \beta_{A,B}) = \delta \circ \gamma_{A,B} = \beta_{A,B}$$

$$(\lambda \circ \delta) \circ \gamma_{A,B} = \lambda \circ (\delta \circ \gamma_{A,B}) = \lambda \circ \beta_{A,B} = \gamma_{A,B}.$$

Therefore by the universal property of $\beta_{A,B}$ it follows $\delta \circ \lambda = \mathbf{I}_{A \otimes_{MV} B}$, and by the universal property of $\gamma_{A,B}$ it follows $\lambda \circ \delta = \mathbf{I}_{A \otimes_\ell B}$, that is the two

tensor product are MV-algebraic isomorphic, i.e. $\Gamma(G_A, u_A) \otimes_{MV} \Gamma(G_B, u_B) \simeq \Gamma(G_A \otimes_{\ell} G_B, u_A \otimes_{\ell} u_B)$. \square

Remark 2.1. We note that, in [14], it is proved that the functor Γ commutes with another tensor product denoted \otimes_o . The tensor product \otimes_o of MV-algebras defined in [14] corresponds to the tensor product \otimes_o of ℓ -groups defined in [18] and it is defined using only bilinear functions, instead of bimorphisms and ℓu -bilinear functions. In this case, if A and B are MV-algebras, the function $\iota_A : A \rightarrow A \otimes_o B$ by $\iota_A(a) = a \otimes_o 1$ is no longer a homomorphism of MV-algebras.

Recall that the functor Γ maps archimedean ℓu -groups to semisimple MV-algebras. Moreover, one can easily prove that Γ also preserve the archimedean tensor product.

Corollary 2.1. *If $(G_A, u_A), \Gamma(G_B, u_B)$ are archimedean ℓu -groups and A, B are semisimple MV-algebras such that $A \simeq \Gamma(G_A, u_A)$ and $B \simeq \Gamma(G_B, u_B)$ then $A \otimes B \simeq \Gamma(G_A \otimes_a G_B, u_A \otimes_a u_B)$.*

Proof. The proof is similar with the one of Theorem 2.1. The main idea is that $\Gamma(G_A \otimes_a G_B, u_A \otimes_a u_B)$ satisfy the same universal property that uniquely defines $A \otimes B$, up to isomorphism. \square

3 Scalar extension property for semisimple MV-algebras

In this section we will investigate the scalar extension property for MV-algebras and ℓu -groups. First, let us state the property in our context.

(SEP_{MV}) If P is a unital PMV-algebra and A is an MV-algebra, then $P \otimes_{MV} A$ has a canonical structure of MV-module over P .

(SEP _{ℓ}) If R is an ℓ -ring and G is an ℓ -group, then $R \otimes_{\ell} G$ has a canonical structure of ℓ -module over R .

We summarize the results so far.

Remark 3.1. The scalar extension property is one of the basic property arising from a tensor product, and while it is straightforward in the non-ordered case, with lattice ordered structures it presents some difficulties.

(1) For MV-algebras, the property SEP_{MV} is stated in [14, Theorem 4.11], but the proof presents a wrong argument. We note that the structure $P \otimes_{MV} A$ can be endowed with a family of unary operations $\{\alpha\}_{\alpha \in P}$ such that the function $(\alpha, x) \mapsto \alpha x$ is linear in the second argument. The proof of the linearity in the first argument contains a mistake.

(2) For ℓ -groups, the property SEP_ℓ is left as an exercise in [23, Chapter 4.5]. The case $R = \mathbb{R}$, i.e. the ℓ -modules are Riesz spaces (vector lattices), is considered in [18, Proposition 2.1], but the details are missing.

Our impossibility to correct the proof from [14, Theorem 4.11] and to complete the proof from [18, Proposition 2.1] is related to the fact that the sum of two homomorphisms of ℓ -groups is not always an homomorphism of ℓ -groups. We leave this as open problems.

Remark 3.2. Under the assumption that [18, Proposition 2.1] is correct, using Theorem 2.1, one can immediately prove (SEP_{MV}) for $P = [0, 1]$. This means that for any MV-algebra A , the tensor product $[0, 1] \otimes_{MV} A$ is a Riesz MV-algebra.

Remark 3.3. In [5, Theorem 5], the property SEP_ℓ is proved for the archimedean tensor product in the context of Riesz spaces ($R = \mathbb{R}$).

In Theorem 3.1 we prove SEP_{MV} for the semisimple tensor product allowing P to be an arbitrary unital and semisimple PMV-algebra. Using Theorem 2.1 we get SEP_ℓ for archimedean structures allowing R to be an arbitrary archimedean ℓ -ring with strong unit. In this way we generalize the result from [5].

Remark 3.4. Let P be a PMV-algebra and $S \subseteq P$ a sub PMV-algebra of P . It is easily seen that P is a S -MV-module such that the external operation coincides with the internal product on P .

Lemma 3.1. *Let $A \subseteq C(X) \subseteq C(X \times Y)$ be a PMV-algebra. Then the map*

$$\varphi : A \times C(X \times Y) \rightarrow C(X \times Y),$$

$$\varphi(a, f)(x, y) = a(x)f(x, y) \text{ for any } a \in A \text{ and } f \in C(X \times Y),$$

defines a structure of A -MV-module.

Proof. If $a_1 + a_2$ is defined in A , then for any $x \in X$ the partial sum $a_1(x) + a_2(x)$ is defined and $(a_1 + a_2)f = a_1f + a_2f$ if and only if the equality holds for any $x \in X, y \in Y$. Trivially $(a_1(x) + a_2(x))f(x, y) = a_1(x)f(x, y) + a_2(x)f(x, y)$,

since the functions are $[0, 1]$ -valued. In the same way we get all other conditions for an MV-module. \square

Theorem 3.1. *Let A be a unital and semisimple PMV-algebra, and B be a semisimple MV-algebra. Then $A \otimes B$ is an A -MV-module.*

Proof. By Theorem 1.1, $A \otimes B = \langle \gamma(a, b) \mid a \in A, b \in B \rangle_{MV} \subseteq C(X \times Y)$, with $A \subseteq C(X)$, $B \subseteq C(Y)$.

For any $\alpha \in A$, we define

$$\omega_\alpha : A \times B \rightarrow A \otimes B, \quad \omega_\alpha(a, b)(x, y) = \alpha(x)a(x)b(y).$$

Since $a(x)b(y) = \gamma(a, b)(x, y)$ and $\alpha(x) \leq 1$, it follows that $\omega_\alpha(a, b) \in A \otimes B$.

We prove that ω_α is a bimorphism.

Note that $\omega_\alpha(a_1 \wedge a_2, b)(x, y) = \alpha(x)[(a_1 \wedge a_2)(x)]b(y) = \alpha(x)(a_1(x) \wedge a_2(x))b(y)$. Since we are in the unital case the product can be distributed on and we get \wedge and \vee $[\alpha(x)a_1(x) \wedge \alpha(x)a_2(x)]b(y) = [\alpha(x)a_1(x)b(y)] \wedge [\alpha(x)a_2(x)b(y)]$. In the same way we get the desired property for \vee and on the second component;

If $a_1 + a_2$ defined in A , then $\omega_\alpha(a_1 + a_2, b)(x, y) = \alpha(x)[a_1(x) + a_2(x)]b(y)$.

By the definition of PMV-algebra, we get the desired conclusion.

Therefore, applying the universal property, there exists a homomorphism of MV-algebras

$$\Omega_\alpha : A \otimes B \rightarrow [0, \omega_\alpha(1, 1)] \leq_i A \otimes B.$$

We remark that $\omega_\alpha(1, 1)(x, y) = \alpha(x)\mathbf{1}(x)\mathbf{1}(y) = \alpha(x)$. Moreover, $\Omega_\alpha : A \otimes B \rightarrow [0, \omega_\alpha(1, 1)] \leq_i A \otimes B \subseteq C(X \times Y)$.

By Lemma 3.1, $C(X \times Y)$ is A -MV-module with external operation $\varphi : A \times C(X \times Y) \rightarrow C(X \times Y)$. We fix $\alpha \in A$, and we define

$$\theta_\alpha : C(X \times Y) \rightarrow C(X \times Y), \quad \theta_\alpha(f) = \varphi(\alpha, f).$$

Since $C(X \times Y)$ is A -MV-module, θ_α is linear and $\theta_\alpha(f) \leq f$. It follows that $\theta_\alpha(f_1) \wedge f_2 = 0$ whenever $f_1 \wedge f_2 = 0$. Note that such functions are called *f-operators* in [15]. By [15, Proposition 5.9], θ_α is an MV-algebra morphism

$$\theta_\alpha : C(X \times Y) \rightarrow [0, \theta_\alpha(1)] \leq_i C(X \times Y).$$

Moreover, $\theta_\alpha(1)(x, y) = \varphi(\alpha, 1)(x, y) = \alpha(x)$.

Then we have

$$\Omega_\alpha : A \otimes B \rightarrow [0, \omega_\alpha(1, 1)] \leq_i A \otimes B \subseteq C(X \times Y) \text{ and}$$

$$\theta_\alpha|_{A \otimes B} : A \otimes B \rightarrow [0, \theta_\alpha(1)] \leq_i C(X \times Y).$$

Since $\omega_\alpha(1, 1) = \theta_\alpha(1)$, by [14, Lemma 2.3],

$$[0, \omega_\alpha(1, 1)]_{A \otimes B} \subseteq [0, \theta_\alpha(1)] \leq_i C(X \times Y).$$

Therefore Ω_α and $\theta_\alpha|_{A \otimes B}$ are both maps from $A \otimes B$ in $[0, \theta_\alpha(1)] \leq_i C(X \times Y)$, and by universal property they are the same map if they coincide on generators, but this is trivially true, since for any $a \in A$, $b \in B$, $x \in X$, $y \in Y$

$$\Omega_\alpha(\gamma(a, b))(x, y) = \alpha(x)a(x)b(y) = \varphi(\alpha, \gamma(a, b))(x, y) = \theta_\alpha(\gamma(a, b))(x, y).$$

As result, we have $A \otimes B$ included in $C(X \times Y)$ as MV-algebra, and two families of linear functions:

$$\{\Omega_\alpha : A \otimes B \rightarrow A \otimes B\}_{\alpha \in A}; \quad \{\theta_\alpha : C(X \times Y) \rightarrow C(X \times Y)\}_{\alpha \in A}$$

such that $\theta_\alpha|_{A \otimes B} = \Omega_\alpha$, and $(C(X \times Y), \{\theta_\alpha\}_{\alpha \in A})$ is an A -MV-module.

Since for any $\alpha \in A$ we have $\theta_\alpha|_{A \otimes B}(A \otimes B) = \Omega_\alpha(A \otimes B) \subseteq A \otimes B$, $A \otimes B$ is closed to the scalar product, and $(A \otimes B, \{\theta_\alpha|_{A \otimes B}\}_{\alpha \in A})$ is an sub MV-module of $C(X \times Y)$. \square

By Theorem 2.1 and Theorem 3.1 we immediately get the following.

Theorem 3.2. *If R is a unital and archimedean ℓu -ring and G is an archimedean ℓu -group, $R \otimes_a G$ is a ℓu -module over R .*

In the sequel, we apply use SEP_{MV} in order to establish connections between MV-algebras and Riesz MV-algebras.

Proposition 3.1. *Let A be a semisimple Riesz MV-algebra, and B be a semisimple MV-algebra. Then $A \otimes B$ is an Riesz MV-algebra. In particular, $[0, 1] \otimes B$ is a Riesz MV-algebra.*

Proof. The proof is similar with the one of Theorem 3.1. The starting bimorphism will be

$$\omega_\alpha : A \times B \rightarrow A \otimes B, \omega_\alpha(a, b) = \alpha a(x)b(y) \text{ where } \alpha \in [0, 1]. \quad \square$$

By Theorem 2.1 and Proposition 3.1 we immediately get the following.

Proposition 3.2. *Let V be an archimedean Riesz Space with strong unit and G be an archimedean ℓu -group. Then $V \otimes_a G$ is an archimedean Riesz Space with strong unit. In particular, $\mathbb{R} \otimes_a G$ is a Riesz Space.*

Remark 3.5. Let B be a semisimple MV-algebra and $\iota_B : B \rightarrow [0, 1] \otimes B$ the canonical embedding. Since $[0, 1] \otimes B$ is generated, as a Riesz MV-algebra, by $\iota_B(B)$ it follows by [12, Corollary 4.2], that $[0, 1] \otimes A$ is, up to isomorphism, the Riesz MV-algebra hull of B . One can see [12] for more details.

Corollary 3.1. *Let B be a semisimple MV-algebra. For any semisimple Riesz MV-algebra V and for any homomorphism of MV-algebras $f : B \rightarrow \mathcal{U}_{\mathbb{R}}(V)$ there is a unique homomorphism of Riesz MV-algebras $\tilde{f} : [0, 1] \otimes B \rightarrow V$ such that $\tilde{f} \circ \iota_B = f$.*

Proof. Define $\beta_f : [0, 1] \times B \rightarrow V$ by $\beta(\alpha, x) = \alpha f(x)$ for any $\alpha \in [0, 1]$ and $x \in B$ and use the universal property of the tensor product. We note that any MV-algebra homomorphism between Riesz MV-algebras preserves the scalar multiplication, so it is a morphism of Riesz MV-algebras [11, Corollary 3.11]. \square

Assume that \mathbf{MV}_{ss} is the full subcategory of semisimple MV-algebras and \mathbf{RMV}_{ss} is the full subcategory of semisimple Riesz MV-algebras. Hence we define a functor $\mathcal{T}_{\otimes} : \mathbf{MV}_{\text{ss}} \rightarrow \mathbf{RMV}_{\text{ss}}$ by

$$\mathcal{T}_{\otimes}(B) = [0, 1] \otimes B \text{ for any semisimple MV-algebra } B \text{ and}$$

$\mathcal{T}_{\otimes}(f) = \tilde{f}$ for any homomorphism of MV-algebras $f : A \rightarrow B$, where $\tilde{f} : [0, 1] \otimes A \rightarrow [0, 1] \otimes B$ is the unique Riesz MV-algebra homomorphism such that $\tilde{f} \circ \iota_A = \iota_B \circ f$ (which exists by Corollary 3.1).

Corollary 3.2. *Under the above hypothesis, $(\mathcal{T}_{\otimes}, \mathcal{U}_{\mathbb{R}})$ is an adjoint pair.*

Proof. It is straightforward. Using a different construction of the Riesz hull, this result is proved in [12]. \square

4 The categorical adjunction between semisimple PMV-algebras and semisimple f MV-algebras

In the sequel, using the scalar extension property, we define an adjunction between the category of semisimple and unital PMV-algebras and the category of unital and semisimple f MV-algebras.

As a preliminary step, we prove the following theorem.

Proposition 4.1. *Let A, B be unital and semisimple PMV-algebras. Then $A \otimes B$ is a unital and semisimple PMV-algebra.*

Proof. By Theorem 1.1, $A \otimes B = \langle \gamma(a, b) \mid a \in A, b \in B \rangle_{MV} \subseteq C(X \times Y)$ with $A \subseteq C(X)$, $B \subseteq C(Y)$, as MV-algebra.

For any $c \in A \otimes B$, we define $\omega_c : A \times B \rightarrow A \otimes B$, $\omega_c(a, b)(x, y) = c(x, y)a(x)b(y)$. Since $a(x)b(y) = \gamma(a, b)(x, y)$ and $c(x, y) \leq 1$, $\omega_c(a, b) \in A \otimes B$ and it is a

bimorphism, likewise in the proof of Theorem 3.1. Therefore, applying the universal property, there exist a map

$$\Omega_c : A \otimes B \rightarrow [0, \omega_c(1, 1)] \leq_i A \otimes B.$$

Again, $\omega_c(1, 1)(x, y) = c(x, y)\mathbf{1}(x)\mathbf{1}(y) = c(x, y)$ and $\Omega_c : A \otimes B \rightarrow [0, \omega_c(1, 1)] \leq_i A \otimes B \subseteq C(X \times Y)$.

It is straightforward that $C(X \times Y)$ is PMV-algebra with internal product $*$: $C(X \times Y) \times C(X \times Y) \rightarrow C(X \times Y)$ defined component-wise. We fix $c \in A \otimes B$, and we define

$$\theta_c : C(X \times X) \rightarrow C(X \times Y), \quad \theta_c(f) = c * f.$$

It is easy to prove, since $C(X \times Y)$ is unital PMV-algebra, that θ_c is linear and $\theta_c(f) \leq f$. And again like in the proof of Theorem 3.1, $\theta_c : C(X \times Y) \rightarrow [0, \theta_c(1)] \leq_i C(X \times Y)$ is an homomorphism of MV-algebras. Moreover, $\theta_c(1)(x, y) = (c * 1)(x, y) = c(x, y)$.

Then we have

$$\Omega_c : A \otimes B \rightarrow [0, \omega_c(1, 1)] \leq_i A \otimes B \subseteq C(X \times Y) \text{ and}$$

$$\theta_c|_{A \otimes B} : A \otimes B \rightarrow [0, \theta_c(1)] \leq_i C(X \times Y).$$

Since $\omega_c(1, 1) = \theta_c(1)$, we get $[0, \omega_c(1, 1)]_{A \otimes B} \subseteq [0, \theta_c(1)] \leq_i C(X \times Y)$, and the conclusion follows like in Theorem 3.1.

Finally, it can be easily seen that the unit in $A \otimes B$ is the element $1_A \otimes 1_B$. Then $A \otimes B$ is unital, therefore it is semisimple as PMV-algebra. \square

Theorem 4.1. *Let R be a fMV-algebra and P be a unital and semisimple PMV-algebra. Then $R \otimes P$ is a unital and semisimple fMV-algebra.*

Proof. By construction $R \otimes P$ is a semisimple MV-algebra, by Corollary 3.1, $R \otimes P$ is a Riesz MV-algebra and by Proposition 4.1 it is a unital and semisimple PMV-algebra. Moreover, by the construction of the product and the scalar operation as the usual product and scalar operation between functions given in Proposition 4.1 and Theorem 3.1, the associativity law between products is satisfied since it holds in any $C(X)$. It follows that $R \otimes P$ is a unital and semisimple fMV-algebra. \square

Proposition 4.2. *Let A be a unital and semisimple PMV-algebra. For any unital and semisimple fMV-algebra M and for any homomorphism of PMV-algebras $f : A \rightarrow \mathcal{U}_{\mathbb{R}}(M)$ there is a unique homomorphism of fMV-algebras $\tilde{f} : [0, 1] \otimes A \rightarrow M$ such that $\tilde{f} \circ \iota_A = f$, where $\iota_A : A \rightarrow [0, 1] \otimes A$ is the embedding in the tensor product.*

Proof. By Corollary 3.1, there exists a homomorphism of Riesz MV-algebras $\tilde{f} : [0, 1] \otimes A \rightarrow \mathcal{U}_{(\cdot)}(M)$ such that $\tilde{f} \circ \iota_A = f$. Since $[0, 1] \otimes A$ is a f MV-algebra by Theorem 4.1, \tilde{f} is a homomorphism of Riesz MV-algebras between unital and semisimple f MV-algebras. By [17, Proposition 3.2] \tilde{f} is a homomorphism of f MV-algebras. \square

Proposition 4.3. *Let P_1 and P_2 be semisimple and unital PMV-algebras, and $h : P_1 \rightarrow P_2$ a homomorphism of PMV-algebras. Then there exists a unique $h^\sharp : [0, 1] \otimes P_1 \rightarrow [0, 1] \otimes P_2$ homomorphism of f MV-algebras such that $h^\sharp \circ \iota_1 = \iota_2 \circ h$, where $\iota_i : P_i \rightarrow [0, 1] \otimes P_i$ for $i = 1, 2$ are the natural embeddings.*

Proof. It is straightforward by Proposition 4.2, with $f = \iota_2 \circ h$. \square

Let $\mathbf{uPMV}_{\text{ss}}$ be the full subcategory of semisimple and unital PMV-algebras with homomorphism of PMV-algebras and let $\mathbf{ufMV}_{\text{ss}}$ be the full subcategory of semisimple and unital f MV-algebras with homomorphism of f MV-algebras. We define a functor $\mathcal{F}_\otimes : \mathbf{uPMV}_{\text{ss}} \rightarrow \mathbf{ufMV}_{\text{ss}}$ as follows

- (i) for any $P \in \mathbf{uPMV}_{\text{ss}}$, $\mathcal{F}_\otimes(P)$ is $[0, 1] \otimes P$. By Theorem 4.1 it is a unital, commutative and semisimple f MV-algebra.
- (ii) for any homomorphism of PMV-algebras $h : P_1 \rightarrow P_2$, $\mathcal{T}(h)$ is the homomorphism of f MV-algebras h^\sharp defined in Proposition 4.3.

From $\mathbf{ufMV}_{\text{ss}}$ to $\mathbf{uPMV}_{\text{ss}}$ we have the usual forgetful functor $\mathcal{U}_{\mathbb{R}}$.

Lemma 4.1. *\mathcal{F}_\otimes is a functor.*

Proof. Let P_1, P_2, P_3 be PMV-algebras and $h : P_1 \rightarrow P_2$ and $g : P_2 \rightarrow P_3$ PMV-algebras homomorphisms. Let ι_1, ι_2 and ι_3 be the embeddings of P_1, P_2, P_3 in $[0, 1] \otimes P_1, [0, 1] \otimes P_2$ and $[0, 1] \otimes P_3$ respectively. Then

$$(g^\sharp \circ h^\sharp) \circ \iota_1 = g^\sharp \circ (h^\sharp \circ \iota_1) = g^\sharp \circ (\iota_2 \circ h) = (g^\sharp \circ \iota_2) \circ h = \iota_3 \circ (g \circ h)$$

and by Proposition 4.3 we get $g^\sharp \circ h^\sharp = (g \circ h)^\sharp$. \square

Lemma 4.2. *The maps $\{\iota_A\}_{A \in \mathbf{uPMV}_{\text{ss}}}$ are a natural transformation between the identity functor on $\mathbf{uPMV}_{\text{ss}}$ and $\mathbf{ufMV}_{\text{ss}}$.*

Proof. Let $P_1, P_2 \in \mathbf{uPMV}_{\text{ss}}$ and let $h : P_1 \rightarrow P_2$ be a homomorphism of PMV-algebras. We need to prove that $\mathcal{U}_{\mathbb{R}}\mathcal{F}_\otimes(h) \circ \iota_1 = \iota_2 \circ h$. Since $\mathcal{U}_{\mathbb{R}}\mathcal{F}_\otimes(h) = h^\sharp$ the result follows from Proposition 4.3. \square

Theorem 4.2. *The functors \mathcal{F}_\otimes and $\mathcal{U}_\mathbb{R}$ are adjoint functors.*

Proof. In order to prove that \mathcal{F}_\otimes is left adjoint functor of $\mathcal{U}_\mathbb{R}$, we need to prove that for any semisimple and unital f MV-algebra A and any homomorphism of PMV-algebras $f : P \rightarrow \mathcal{U}_\mathbb{R}(A)$, with $P \in \mathbf{uPMV}_{\text{ss}}$, there exists a homomorphism of f MV-algebras $f^\sharp : \mathcal{T}(P) \rightarrow A$ such that $\mathcal{U}_\mathbb{R}(f^\sharp) \circ \iota_P = f$. This follows from Proposition 4.2. \square

Results in the previous sections can be transferred to ℓu -groups and all related structures. We remark that Proposition 4.1, Theorem 4.1 and Corollary 2.1 entail the following.

Proposition 4.4. *(i) If R and S are unital and archimedean ℓu -rings, $R \otimes_a S$ is a unital and archimedean ℓu -ring.*

(ii) If V is a unital and archimedean fu -algebra and R is a unital and archimedean ℓu -ring, $V \otimes_a R$ is a unital and archimedean fu -algebra.

Proof. It is straightforward by Theorem 4.1, Corollary 2.1 and the categorical equivalence. \square

5 Conclusions

By categorical equivalence, the adjunctions $(\mathcal{T}_\otimes, \mathcal{U}_\mathbb{R})$ and $(\mathcal{F}_\otimes, \mathcal{U}_\mathbb{R})$ naturally transfer to lattice-ordered structure. We denote by \mathbf{auG}_a the category of archimedean ℓu -groups; \mathbf{uR}_a the category of archimedean and unital ℓu -rings; \mathbf{uRS}_a the category of archimedean Riesz Spaces with strong unit; \mathbf{fuAlg}_a the category of archimedean and unital fu -algebras.

Applying the inverse of Γ and $\Gamma_\mathbb{R}$, $(\mathcal{T}_\otimes, \mathcal{U}_\mathbb{R})$ extends to $(\mathcal{T}_{\otimes a}, \mathcal{U}_{\ell\mathbb{R}})$. This is an adjunction between \mathbf{auG}_a and \mathbf{uRS}_a .

Applying the converses of the functors $\Gamma_{(\cdot)}$ and Γ_f , $(\mathcal{F}_\otimes, \mathcal{U}_\mathbb{R})$ extends to $(\mathcal{F}_{\otimes a}, \mathcal{U}_{\ell\mathbb{R}})$. This is an adjunction between \mathbf{uR}_a and \mathbf{fuAlg}_a .

In the following we summarize our results.

Theorem 5.1. *The following diagrams are commutative:*

$$\begin{array}{ccc}
& \xleftarrow{\mathcal{F}_{\otimes_a}} & \\
\text{fuAlg}_a & \xrightarrow{\mathcal{U}_{(\ell\mathbb{R})}} & \text{uR}_a \\
\downarrow \Gamma_f & & \downarrow \Gamma_{(\cdot)} \\
\text{fMV}_{ss} & \xrightarrow{\mathcal{U}_{\mathbb{R}}} & \text{PMV}_{ss} \\
& \xleftarrow{\mathcal{F}_{\otimes}} &
\end{array}
\qquad
\begin{array}{ccc}
& \xleftarrow{\mathcal{T}_{\otimes_a}} & \\
\text{auG}_a & \xrightarrow{\mathcal{U}_{(\ell\mathbb{R})}} & \text{uRS}_a \\
\downarrow \Gamma & & \downarrow \Gamma_{\mathbb{R}} \\
\text{MV}_{ss} & \xrightarrow{\mathcal{U}_{\mathbb{R}}} & \text{RMV}_{ss} \\
& \xleftarrow{\mathcal{T}_{\otimes}} &
\end{array}$$

Proof. It is a direct consequence of Theorem 2.1. \square

Recall that MV-algebras were defined as the algebraic structures corresponding to Łukasiewicz ∞ -valued logic. Even if their algebraic theory is relevant in itself as proved by [8, 22], it has been developed in strong connection with the associated logical system. The same holds for PMV-algebras [19], their theory has its origins in the problem of enriching Łukasiewicz logic with a binary conector whose interpretation in $[0, 1]$ is the natural product. Logical systems were also defined for Riesz MV-algebras [11] and f MV-algebras [17]. Note that for PMV-algebras and f MV-algebras the logical systems are developed only for particular suitable subclasses. One important link between logic and algebra in all these cases is the Lindenbaum-Tarski algebra, which is the free algebra generated by the set of propositional variables. Due to the fact that the free algebras are semisimple structures, the scalar extension property allows us to connect the free structures as in Proposition 5.1.

For a nonempty set X , let $Free_{MV}(X)$ and $Free_{RMV}(X)$ be the free MV-algebra and, respectively, the free Riesz MV-algebras generated by X . Let $Free_{PMV}(X)$ be the free PMV-algebra in $HSP([0, 1]_{PMV})$, the variety of PMV-algebras generated by $[0, 1]$. Similarly, let $Free_{fMV}(X)$ be the free f MV-algebra in $HSP([0, 1]_{fMV})$, the variety of f MV-algebras generated by $[0, 1]$. See more details in [8, 19, 11, 17].

Proposition 5.1. *For any nonempty set X , the following hold:*

- (i) $Free_{RMV}(X) \simeq [0, 1] \otimes Free_{MV}(X)$,
- (ii) $Free_{fMV}(X) \simeq [0, 1] \otimes Free_{PMV}(X)$.

Proof. (i) follows by Remark 3.5 and [12, Proposition 4.1]; it can also be proved directly, similarly with (ii).

(ii) Assume V is a f MV-algebra and $f : X \rightarrow V$ is a function. Hence there is a unique homomorphism of PMV-algebras $f^\# : Free_{PMV}(X) \rightarrow \mathcal{U}_{\mathbb{R}}(V)$ which extends f . By Proposition 4.2, there exists a homomorphism of f MV-algebras $\tilde{f} : [0, 1] \otimes Free_{PMV}(X) \rightarrow V$ such that $\tilde{f} \circ \iota_{Free_{PMV}(X)} = f^\#$, so $\tilde{f}(1 \otimes x) = f(x)$ for any $x \in X$. The uniqueness of \tilde{f} is a consequence of the uniqueness of $f^\#$. Since $\iota_{Free_{PMV}(X)}$ is an embedding we have $X \simeq \{1 \otimes x \mid x \in X\}$ so $[0, 1] \otimes Free_{PMV}(X)$ satisfies the universal property that uniquely defines $Free_{fMV}(X)$. \square

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